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2-variable Laguerre matrix polynomials and Lie-algebraic techniques*

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Abstract
The authors introduce 2-variable forms of Laguerre and modified Laguerre matrix polynomials and derive their special properties. Further, the representations of the special linear Lie algebra $sl(2)$ and the harmonic oscillator Lie algebra $\mathcal{G}(0, 1)$ are used to derive certain results involving these polynomials. Furthermore, the generating relations for the ordinary as well as matrix polynomials related to these matrix polynomials are derived as applications.

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1. Introduction

Special matrix functions appear in the literature related to statistics [7] and recently in connection with matrix analogues of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families [8–10].

Jódar *et al* [9] considered systems of second-order differential equations of the form

$$xX''(x) + (A + I - \lambda xI)X'(x) + CX(x) = 0, \tag{1.1}$$

where λ is a complex number, x is a real number and A, C and $X(x) \in \mathbb{C}^{m\times m}$.

Equation (1.1) is important from the viewpoint of applications, since the systems of second-order differential equations of the type

$$A(x)X''(x) + B(x)X'(x) + C(x)X(x) = 0, \tag{1.2}$$

where $A(x)$, $B(x)$ and $C(x)$ are matrix-valued functions, occur frequently in physics, chemistry and mechanics [12, 19, 20]. Such systems also appear when one applies semi discretization techniques to solve partial differential equations [21].

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Orthogonal matrix polynomials have become more and more relevant in the last two decades. Laguerre matrix polynomials (LM$_a$P) $L_n^{(A,\lambda)}(x)$ have been introduced and studied by Jódar *et al* [9].

If A is a matrix in $\mathbb{C}^{m\times m}$ such that the following spectral condition is satisfied:

$$-k \notin \sigma(A) \text{for every integer } k > 0 \tag{1.3}$$

and λ is a complex number with $\mathrm{Re}(\lambda) > 0$, then the nth Laguerre matrix polynomial are defined by [9; p 58(3.7)]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}x^k, \qquad n \geqslant 0, \tag{1.4}$$

where $(A)_n$ is the matrix Pochhammer symbol defined by

$$(A)_n = A(A+I)\cdots(A+(n-1)I), \qquad n \geqslant 1; \quad (A)_0 = I.$$

The LM$_a$P $L_n^{(A,\lambda)}(x)$ is a solution of the differential equation

$$xX''(x) + (A + I - \lambda x I)X'(x) + \lambda n X(x) = 0, \tag{1.5}$$

and the generating function of LM$_a$P is given by [9; p 57]

$$(1-t)^{-(A+I)} \exp\left(\frac{-\lambda x t}{(1-t)}\right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n, \qquad x, t \in \mathbb{C}; \quad |t| < 1. \tag{1.6}$$

Many special functions and also some elementary functions are special cases of the hypergeometric functions. Jódar and Cortés [11] introduced the Gauss hypergeometric matrix function ${}_2F_1[A, B; C; z]$ as a matrix power series in the form

$${}_2F_1[A, B; C; z] = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n (C)_n^{-1}}{n!} z^n, \tag{1.7}$$

where A, B, C are matrices in $\mathbb{C}^{m\times m}$ such that

$$C + nI \text{ is invertible for all integers } n \geqslant 0. \tag{1.8}$$

The hypergeometric matrix function ${}_2F_1[A, B; C; z]$ converges when $|z| < 1$. Also, if A, B, C are positive stable matrices in $\mathbb{C}^{m\times m}$ such that

$$\beta(C) > \alpha(A) + \alpha(B), \tag{1.9}$$

then the series (1.7) is absolutely convergent for $|z| = 1$.

Further, if C is a matrix in $\mathbb{C}^{m\times m}$ satisfying (1.8) and $CB = BC$, then ${}_2F_1[A, B; C; z]$ is the solution of the matrix differential equation [11; p 211(25)]

$$z(1-z)W''(z) - zAW'(z) + W'(z)(C - z(B+I)) - AW(z)B = 0, \qquad 0 \leqslant z \leqslant 1 \tag{1.10}$$

satisfying ${}_2F_1[A, B; C; 0] = I$.

The 2-variable Laguerre polynomials have been introduced and studied in [3–5]. These polynomials are shown to be the natural solutions of a particular set of partial differential equations, which often appears in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings [24].

The 2-variable associated Laguerre polynomials (2VALP) are specified by the series [3; p 113(14)]

$$L_n^{(a)}(x, y) = \sum_{k=0}^{n} \frac{(a+1)_n (-1)^k x^k y^{n-k}}{(a+1)_k (n-k)! k!}, \tag{1.11}$$

and the generating function for $L_n^{(a)}(x, y)$ is given as [3; p 113(15)]

$$(1 - yt)^{-a-1} \exp\left(\frac{-xt}{(1 - yt)}\right) = \sum_{n=0}^{\infty} L_n^{(a)}(x, y)t^n. \tag{1.12}$$

Furthermore, even though there exists the following close relationship:

$$L_n^{(a)}(x, y) = y^n L_n^{(a)}\left(\frac{x}{y}\right), \tag{1.13}$$

with the associated Laguerre polynomials $L_n^{(a)}(x)$ [1], yet the usage of a second variable in the 2VALP $L_n^{(a)}(x, y)$ is found to be convenient from the viewpoint of their applications.

Motivated by the work of Jódar and his co-authors on Laguerre matrix polynomials [9] and due to the importance of 2-variable forms of Laguerre polynomials [3–5], in this paper, we introduce 2-variable Laguerre and modified Laguerre matrix polynomials. Further, the contributions of Khan and her co-workers, see [13–15], related to Lie-theoretic generating relations of Hermite and Laguerre polynomials motivated us to derive generating relations involving these matrix polynomials by using Lie-algebraic techniques.

In section 2, we give the definition and properties of 2-variable Laguerre and modified Laguerre matrix polynomials. In section 3, we derive the generating relations involving 2-variable Laguerre matrix polynomials ($2VLM_aP$) $L_n^{(A,\lambda)}(x, y)$ by constructing a three-dimensional Lie algebra isomorphic to special linear algebra $sl(2)$, by using Weisner's [23] group-theoretic approach. In section 4, we use the representation theory of the Lie algebra $\mathcal{G}(0, 1)$, to derive generating relations involving 2-variable modified Laguerre matrix polynomials ($2VMLM_aP$) $f_n^{(A,\lambda)}(x, y)$. In section 5, we discuss certain special cases which would yield inevitably many new and known generating relations for the polynomials related to $2VLM_aP$ $L_n^{(A,\lambda)}(x, y)$ and $2VMLM_aP$ $f_n^{(A,\lambda)}(x, y)$. Finally, we give some concluding remarks in section 6.

2. 2-variable Laguerre and modified Laguerre matrix polynomials

In view of the equations (1.6), (1.12) and (1.13), the generating function for 2-variable Laguerre matrix polynomials $2VLM_aP$ $L_n^{(A,\lambda)}(x, y)$ can be cast in the form

$$(1 - yt)^{-(A+I)} \exp\left(\frac{-\lambda xt}{(1 - yt)}\right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x, y)t^n, \tag{2.1}$$

where A is a matrix in $\mathbb{C}^{m\times m}$ and λ is a complex number with $\mathrm{Re}(\lambda) > 0$. The generating function (2.1) is defined for complex values of x, y and t with $|yt| < 1$.

In order to obtain the series definition for $2VLM_aP$, we consider the matrix valued function

$$G_A(x, y; t) = (1 - yt)^{-(A+I)} \exp\left(\frac{-\lambda xt}{(1 - yt)}\right). \tag{2.2}$$

Further, in view of equation (2.1) and due to the fact that $G_A(x, y; t)$ is a function of the complex variable t, which is holomorphic in $|yt| < 1$, we can represent $G_A(x, y; t)$ by a power series at $t = 0$ of the form

$$G_A(x, y; t) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x, y)t^n, \tag{2.3}$$

which on using equation (2.2) becomes

$$G_A(x, y; t) = (1 - yt)^{-(A+I)} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k x^k t^k}{k!(1 - yt)^k}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k x^k t^k}{k!} (1 - yt)^{-(A+(k+1)I)}. \tag{2.4}$$

If we consider the Taylor expansion of the function $(1 - yt)^{-(A+(k+1)I)}$ at $t = 0$, we can write

$$(1 - yt)^{-(A+(k+1)I)} = \sum_{n=0}^{\infty} (A + I)_{n+k} ((A + I)_k)^{-1} \frac{y^n t^n}{n!}, \tag{2.5}$$

such that

$$A + nI \text{ is an invertible for every integer } n \geqslant 0. \tag{2.6}$$

From equations (2.4) and (2.5), it follows that

$$G_A(x, y; t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k \lambda^k x^k y^{n-k} t^n}{k!(n - k)!} (A + I)_n ((A + I)_k)^{-1}. \tag{2.7}$$

Thus, from equations (2.3) and (2.7) and by identification of the coefficient of t^n, we obtain the following series definition for the 2VLM$_a$P $L_n^{(A,\lambda)}(x, y)$:

$$L_n^{(A,\lambda)}(x, y) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k x^k y^{n-k}}{k!(n - k)!} (A + I)_n ((A + I)_k)^{-1}, \tag{2.8}$$

which can also be expressed in terms of the confluent hypergeometric function ${}_1F_1$ [1] as

$$L_n^{(A,\lambda)}(x, y) = \frac{\Gamma(A + (n + 1)I)(\Gamma(A + I))^{-1} y^n}{\Gamma(n + 1)} \, {}_1F_1\Big[- n; A + I; \frac{\lambda x}{y} \Big]. \tag{2.9}$$

Now, by differentiating the generating function (2.1) with respect to t, x and y, we obtain the following pure and differential matrix recurrence relations satisfied by 2VLM$_a$P $L_n^{(A,\lambda)}(x, y)$:

$$(n + 1)L_{n+1}^{(A,\lambda)}(x, y) - (yA - (\lambda x - y(2n + 1))I)L_n^{(A,\lambda)}(x, y)$$
$$+ y^2(A + nI)L_{n-1}^{(A,\lambda)}(x, y) = 0, \qquad n \geqslant 1 \tag{2.10}$$

and

$$\frac{\partial}{\partial x} L_n^{(A,\lambda)}(x, y) = \frac{1}{x}\big\{nL_n^{(A,\lambda)}(x, y) - y(A + nI)L_{n-1}^{(A,\lambda)}(x, y)\big\},$$
$$\frac{\partial}{\partial x} L_n^{(A,\lambda)}(x, y) = \frac{1}{xy}\big\{(n + 1)L_{n+1}^{(A,\lambda)}(x, y) - (yA - (\lambda x - y(n + 1))I)L_n^{(A,\lambda)}(x, y)\big\}, \tag{2.11}$$
$$\frac{\partial}{\partial y} L_n^{(A,\lambda)}(x, y) = (A + nI)L_{n-1}^{(A,\lambda)}(x, y).$$

From these recurrence relations, we conclude that the 2VLM$_a$P $L_n^{(A,\lambda)}(x, y)$ are the solutions of the following matrix differential equation:

$$\left(x\frac{d^2}{dx^2} + \left(A + \left(1 - \frac{\lambda x}{y} \right) I \right) \frac{d}{dx} + \frac{\lambda n}{y} \right) L_n^{(A,\lambda)}(x, y) = 0. \tag{2.12}$$

Further, we introduce the 2VMLM$_a$P $f_n^{(A,\lambda)}(x, y) = (-1)^n L_n^{(-A-nI,\lambda)}(x, y)$. First we derive the generating function of 2-variable modified Laguerre polynomials (2VMLP)

$f_n^\beta(x,y) = (-1)^n L_n^{(-\beta-n)}(x,y)$ with the help of Brown's method [2; pp. 32–40] of finding pairs of generating functions by using an auxiliary parameter, see also [17; p 80].

We choose two sets $\left\{L_n^{(a)}(x,y)\right\}$ and $\left\{(-1)^n L_n^{(-a-n)}(x,y)\right\}$, which have a common parameter a. Suppose that by introducing a second parameter m, we are able to construct a set $\left\{P_n^{(a)}(m;x,y)\right\}$ such that the choice given sets are special cases of the constructed set.

If we are able to find a generating function for the constructed set $\left\{P_n^{(a)}(m;x,y)\right\}$, then it is possible to find a pair of generating functions for the two given sets. We recall that 2VALP $L_n^{(a)}(x,y)$ are defined by the series (1.11) and the 2-variable modified Laguerre polynomials (2VMLP) are defined by the series

$$(-1)^n L_n^{(-a-n)}(x,y) = \sum_{k=0}^{n} \frac{(a)_{n-k} x^k y^{n-k}}{k!(n-k)!}. \tag{2.13}$$

By examining the expansions (1.11) and (2.13), we are able to construct a generalization $P_n^a(m;x,y)$ in the following form:

$$P_n^a(m;x,y) = \sum_{k=0}^{n} \frac{(a+1+mk)_{n-k}(-1)^{mk} x^k y^{n-k}}{k!(n-k)!}, \tag{2.14}$$

where m is a nonnegative integer. Then

$$P_n^a(1;x,y) = L_n^{(a)}(x,y) \tag{2.15}$$

and

$$P_n^a(0;x,y) = (-1)^n L_n^{(-a-n)}(x,y). \tag{2.16}$$

We use direct summation techniques to obtain a generating function for the set $\left\{P_n^a(m;x,y)\right\}$. Using result [22; p 100(2)] in equation (2.14), we have

$$\sum_{n=0}^{\infty} P_n^a(m;x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a+1+mk)_n(-1)^{mk} x^k y^n t^{n+k}}{k!n!}.$$

Thus, finally we obtain

$$\sum_{n=0}^{\infty} P_n^a(m;x,y)t^n = (1-yt)^{-a-1} \exp\left(\frac{xt}{(yt-1)^m}\right). \tag{2.17}$$

Further, using equations (2.15) and (2.16) in equation (2.17) we obtain the pair of generating functions as (1.12) and

$$(1-yt)^{-a}\exp(xt) = \sum_{n=0}^{\infty}(-1)^n L_n^{(-a-n)}(x,y)t^n.$$

Thus, we conclude that the 2VMLP $f_n^\beta(x,y) = (-1)^n L_n^{(-\beta-n)}(x,y)$ are defined by the generating function

$$(1-yt)^{-\beta}\exp(xt) = \sum_{n=0}^{\infty} f_n^\beta(x,y)t^n, \tag{2.18}$$

with the series definition

$$f_n^\beta(x,y) = \sum_{k=0}^{n} \frac{(\beta)_{n-k} x^k y^{n-k}}{k!(n-k)!}, \qquad n = 0,1,2,\ldots. \tag{2.19}$$

Now, we are able to introduce 2VMLM$_a$P $f_n^{(A,\lambda)}(x,y)$, where A is a complex matrix satisfying condition (1.3) and λ is a complex number such that $\mathrm{Re}(\lambda) > 0$. These polynomials have 2VMLP as a scalar case. By recalling the generating function (2.18) of 2VMLP $f_n^{\beta}(x, y)$ and from the properties of the matrix functional calculus [6] we establish the generating function of 2VMLM$_{\mathrm{a}}$P $f_n^{(A,\lambda)}(x, y)$ as

$$(1 - yt)^{-A}\exp(\lambda xt) = \sum_{n=0}^{\infty} f_n^{(A,\lambda)}(x, y)t^n. \tag{2.20}$$

We note for $A = \beta \in \mathbb{C}^{1\times 1}$ and $\lambda = 1$ the generating function (2.20) reduces to the scalar case (2.18). Now, we will use the matrix-valued functions (2.20) to show that the series expansion of the 2VMLM$_{\mathrm{a}}$P $f_n^{(A,\lambda)}(x, y)$ is given by

$$f_n^{(A,\lambda)}(x, y) = \sum_{k=0}^{n} \frac{(A)_{n-k}(\lambda x)^k y^{n-k}}{k!(n-k)!}, \qquad n = 0, 1, 2, \ldots. \tag{2.21}$$

We assume that A is a matrix in $\mathbb{C}^{m\times m}$ satisfying condition (1.3). Let λ be a complex number whose real part is positive and consider the matrix-valued function

$$F_A(x, y; t) = (1 - yt)^{-A}\exp(\lambda xt), \tag{2.22}$$

defined for complex values of x, y and t with $|yt| < 1$.

Note that $F_A(x, y; t)$, regarded as a function of the complex variable t is holomorphic in $|yt| < 1$ and therefore F_A is representable by a power series at $t = 0$ of the form

$$F_A(x, y; t) = \sum_{n=0}^{\infty} f_n^{(A,\lambda)}(x, y)t^n. \tag{2.23}$$

From equation (2.22), we have

$$F_A(x, y; t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(A)_{n-k}(\lambda x)^k y^{n-k}t^n}{k!(n-k)!}. \tag{2.24}$$

Finally, by using equations (2.23) and (2.24), we obtain the series definition (2.21). Also, making use of the generating function (2.20), we derive the following pure and differential recurrence relations satisfied by 2VMLM$_{\mathrm{a}}$P $f_n^{(A,\lambda)}(x, y)$:

$$(n+1)f_{n+1}^{(A,\lambda)}(x, y) - (yA + (\lambda x + ny)I)f_n^{(A,\lambda)}(x, y) + \lambda xy f_{n-1}^{(A,\lambda)}(x, y) = 0 \tag{2.25}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial x} f_n^{(A,\lambda)}(x, y) = \lambda f_{n-1}^{(A,\lambda)}(x, y),\\
&\frac{\partial}{\partial x} f_n^{(A,\lambda)}(x, y) = \frac{1}{xy}\big((yA + (\lambda x + ny)I)f_n^{(A,\lambda)}(x, y) - (n+1)f_{n+1}^{(A,\lambda)}(x, y)\big),\\
&\frac{\partial}{\partial y} f_n^{(A,\lambda)}(x, y) = \frac{1}{y}\big(nf_n^{(A,\lambda)}(x, y) - \lambda x f_{n-1}^{(A,\lambda)}(x, y)\big),
\end{aligned} \tag{2.26}$$

respectively.

From these recurrence relations, we conclude that the 2VMLM$_{\mathrm{a}}$P $f_n^{(A,\lambda)}(x, y)$ are the solutions of the following differential equation:

$$xy\frac{\partial^2}{\partial x^2} f_n^{(A,\lambda)}(x, y) - (yA + (\lambda x - y(1-n))I)\frac{\partial}{\partial x} f_n^{(A,\lambda)}(x, y) + \lambda n f_n^{(A,\lambda)}(x, y) = 0. \tag{2.27}$$

3. Group-theoretic method and Laguerre matrix polynomials

In order to make use of the Lie group-theoretic method, first we construct a matrix partial differential equation corresponding to the matrix differential equation (2.12) of 2VLM$_\mathrm{a}$P $L_n^{(A,\lambda)}(x, y)$. Further, using the recurrence relations (2.10) and (2.11), we determine the first-order linear differential operators which form the base to compute the multiplier representation [18; p 17] of the complex special linear Lie group $SL(2)$.

We replace $\frac{\mathrm{d}}{\mathrm{d}x}$ by $\frac{\partial}{\partial x}$, n by $t\frac{\partial}{\partial t}$ and $L_n^{(A,\lambda)}(x, y)$ by $f(x, y; t)$ in equation (2.12), to construct the following partial differential equation:

$$\left(x\frac{\partial^2}{\partial x^2} + \left(A + \left(1 - \frac{\lambda x}{y}\right) I\right)\frac{\partial}{\partial x} + \frac{\lambda t}{y}\right) f(x, y; t) = 0. \tag{3.1}$$

Therefore, $f(x, y; t) = L_n^{(A,\lambda)}(x, y)t^n$ is a solution of equation (3.1), since $L_n^{(A,\lambda)}(x, y)$ is a solution of equation (2.12).

Next, using the recurrence relations (2.10) and (2.11), we determine the following linear partial differential operators:

$$\begin{aligned}
J^3 &= t\frac{\partial}{\partial t} + \frac{1}{2}(A + I),\\
J^+ &= xyt\frac{\partial}{\partial x} + yt^2\frac{\partial}{\partial t} + (yA + (y - \lambda x)I)t,\\
J^- &= \frac{x}{yt}\frac{\partial}{\partial x} - \frac{1}{y}\frac{\partial}{\partial t},
\end{aligned} \tag{3.2}$$

such that

$$\begin{aligned}
J^+\left[L_n^{(A,\lambda)}(x, y)t^n\right] &= (n + 1)L_{n+1}^{(A,\lambda)}(x, y)t^{n+1},\\
J^-\left[L_n^{(A,\lambda)}(x, y)t^n\right] &= -(A + nI)L_{n-1}^{(A,\lambda)}(x, y)t^{n-1},\\
J^3\left[L_n^{(A,\lambda)}(x, y)t^n\right] &= \left(\frac{1}{2}A + \left(n + \frac{1}{2}\right) I\right) L_n^{(A,\lambda)}(x, y)t^n.
\end{aligned} \tag{3.3}$$

The operators J^3, J^+ and J^- are linearly independent operators, defined on $\mathcal{F}$, the complex space of all functions analytic in some neighbourhood of $(x^0, y^0, t^0) \in \mathbb{C}^3$. We observe that the operators J^3, J^+ and J^- given in equation (3.2) satisfy the following commutation relations:

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^3. \tag{3.4}$$

These commutation relations are identical with the commutation relations satisfied by the basis elements [18; p 7(1.18)] of the special linear algebra $sl(2)$, the Lie algebra of the Lie group $SL(2)$. Thus, we conclude that the *J*-operators J^3, J^+ and J^- generate a three-dimensional Lie algebra isomorphic to $sl(2)$.

In terms of the *J*-operators, we introduce the Casimir operator [18; p 32]

$$\begin{aligned}
\mathcal{C} &= J^+J^- + J^3J^3 - J^3\\
&= x\left(x\frac{\partial^2}{\partial x^2} + \left(A + \left(1 - \frac{\lambda x}{y}\right)I\right)\frac{\partial}{\partial x} + \frac{\lambda t}{y}\frac{\partial}{\partial t}\right) + \frac{1}{4}(A^2 - I).
\end{aligned} \tag{3.5}$$

It is easy to verify that the *J*-operators commute with the Casimir operator $\mathcal{C}$, that is

$$[\mathcal{C}, J^3] = [\mathcal{C}, J^\pm] = 0. \tag{3.6}$$

Expression (3.5) enables us to write equation (3.1) as

$$\mathcal{C}f(x, y; t) = \tfrac{1}{4}(A^2 - I)f(x, y; t). \tag{3.7}$$

Now, we proceed to compute the multiplier representation $[T(g)f](x, y; t)$, $g \in SL(2)$ induced by the J-operators (3.2). First we compute the actions of $\exp(b'\mathcal{J}^+)$, $\exp(c'\mathcal{J}^-)$ and $\exp(\tau'\mathcal{J}^3)$ on $f(x, y; t)$, where $\mathcal{J}^+$, $\mathcal{J}^-$ and $\mathcal{J}^3$ are the basis elements of the Lie algebra $sl(2)$.

Using [18; p 18 (theorem 1.10)] and equations (3.2), we find

$$\begin{aligned}
[T(\exp(b'\mathcal{J}^+))f](x, y; t) &= (1 - b'yt)^{-(A+I)} \\
&\quad \times \exp\left(\frac{-b'\lambda xt}{(1 - b'yt)}\right) f\left(\frac{x}{(1 - b'yt)}, y; \frac{t}{(1 - b'yt)}\right), \qquad |b'yt| < 1, \\
[T(\exp(c'\mathcal{J}^-))f](x, y; t) &= f\left(\frac{x}{\left(1 - \frac{c'}{yt}\right)}, y; t\left(1 - \frac{c'}{yt}\right)\right), \qquad \left|\frac{c'}{yt}\right| < 1, \\
[T(\exp(\tau'\mathcal{J}^3))f](x, y; t) &= \exp\left(\frac{1}{2}(A + I)\tau'\right) f(x, y; te^{\tau'}),
\end{aligned} \tag{3.8}$$

defined for $|b'|$, $|c'|$ and $|\tau'|$ sufficiently small, where b', c' and τ' are arbitrary constants and $f(x, y; t)$ is an arbitrary function.

For $g \in SL(2)$ and $d \neq 0$, it is a straightforward computation to show that

$$g = \exp(b'\mathcal{J}^+)\exp(c'\mathcal{J}^-)\exp(\tau'\mathcal{J}^3),$$

where $b' = -\frac{b}{d}$, $c' = -cd$, $\exp\left(\frac{\tau'}{2}\right) = \frac{1}{d}$, $0 \leqslant Im\tau' < 4\pi$ and $ad - bc = 1$.

Hence, the operator $T(g)$ is given by

$$\begin{aligned}
[T(g)f](x, y; t) &= (byt + d)^{-(A+I)} \exp\left(\frac{b\lambda xt}{(byt + d)}\right) f\left(\frac{xyt}{(ayt + c)(byt + d)}, y; \frac{(ayt + c)}{y(byt + d)}\right), \\
&\max\left\{\left|\frac{byt}{d}\right|, \left|\frac{c}{ayt}\right|\right\} < 1; \quad |\arg(d)| < \pi.
\end{aligned} \tag{3.9}$$

To accomplish our task of obtaining generating relations, we search for the matrix function $f(x, y; t)$ which satisfies equation (3.7). Consider the case when $f(x, y; t)$ is a common eigenfunction of $\mathcal{C}$ and J^3, that is, let $f(x, y; t)$ be a solution of simultaneous equations

$$\begin{aligned}
\mathcal{C}f(x, y; t) &= \tfrac{1}{4}(A^2 - I)f(x, y; t), \\
J^3 f(x, y; t) &= \left(\tfrac{1}{2}A + \left(\nu + \tfrac{1}{2}\right)I\right)f(x, y; t),
\end{aligned} \tag{3.10}$$

which may be rewritten as

$$\begin{aligned}
&\left(x\frac{\partial^2}{\partial x^2} + \left(A + \left(1 - \frac{\lambda x}{y}\right)I\right)\frac{\partial}{\partial x} + \frac{\lambda t}{y}\frac{\partial}{\partial t}\right) f(x, y; t) = 0, \\
&\left(t\frac{\partial}{\partial t} - \nu\right) f(x, y; t) = 0.
\end{aligned} \tag{3.11}$$

Equations (3.11) yield

$$f(x, y; t) = L_\nu^{(A,\lambda)}(x, y)t^\nu,$$

so that, we have

$$\begin{aligned}
[T(g)f](x, y; t) &= (byt + d)^{-(A+(\nu+1)I)}\left(\frac{(ayt + c)}{y}\right)^\nu \exp\left(\frac{b\lambda xt}{(byt + d)}\right) \\
&\quad \times L_\nu^{(A,\lambda)}\left(\frac{xyt}{(ayt + c)(byt + d)}, y\right),
\end{aligned} \tag{3.12}$$

satisfying the relation

$$\mathcal{C}[T(g)f](x, y; t) = \tfrac{1}{4}(A^2 - I)[T(g)f](x, y; t).$$

If v is not an integer, then equation (3.12) has an expansion of the form

$$[T(g)f](x,y;t)=\sum_{n=-\infty}^{\infty}J_n(g)L_{n+\nu}^{(A,\lambda)}(x,y)t^{\nu+n}. \tag{3.13}$$

Therefore, we prove the following result:

Theorem 3.1. *The following generating equation holds:*

$$\begin{aligned}&\left(1+\frac{byt}{d}\right)^{-(A+(\nu+1)I)}\left(1+\frac{c}{ayt}\right)^{\nu}\exp\left(\frac{b\lambda xt}{(byt+d)}\right)L_{\nu}^{(A,\lambda)}\left(\frac{xyt}{(ayt+c)(byt+d)},y\right)\\&\quad=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+\nu+n)}{\Gamma(1+\nu)}\left(\frac{-byt}{d}\right)^{n}L_{n+\nu}^{(A,\lambda)}(x,y)\{\Gamma(1+n)\}^{-1}\\&\quad\times{}_2F_1\left[-\nu,A+(\nu+n+1)I;n+1;\frac{bc}{ad}\right],\\&\quad\left|\frac{byt}{d}\right|<1;\quad\left|\frac{c}{ayt}\right|<1;\quad ad-bc=1.\end{aligned} \tag{3.14}$$

Proof. Using equations (3.12) and (3.13), we obtain

$$\begin{aligned}&(byt+d)^{-(A+(\nu+1)I)}\left(\frac{(ayt+c)}{y}\right)^{\nu}\exp\left(\frac{b\lambda xt}{(byt+d)}\right)L_{\nu}^{(A,\lambda)}\left(\frac{xyt}{(ayt+c)(byt+d)},y\right)\\&\quad=\sum_{n=-\infty}^{\infty}J_n(g)L_{n+\nu}^{(A,\lambda)}(x,y)t^{n}.\end{aligned} \tag{3.15}$$

We determine $J_n(g)$, by setting $x=0$ in equation (3.15) to obtain

$$\begin{aligned}J_n(g)&=(-1)^n\Gamma(1+\nu+n)a^{\nu}b^{n}d^{-(A+(\nu+n+1)I}(\Gamma(1+\nu))^{-1}(\Gamma(1+n))^{-1}\\&\quad\times{}_2F_1\left[-\nu,A+(\nu+n+1)I;n+1;\frac{bc}{ad}\right].\end{aligned} \tag{3.16}$$

where ${}_2F_1$ denotes the hypergeometric matrix function defined by equation (1.7).

Finally, substituting the expression for $J_n(g)$ given by equation (3.16) into equation (3.15), we obtain result (3.14). □

Remark 1. The following corollary is an immediate consequence of theorem 3.1, when ν is a nonnegative integer, say $\nu=k$.

Corollary 1. *The following generating equation holds:*

$$\begin{aligned}&\left(1+\frac{byt}{d}\right)^{-(A+(k+1)I)}\left(1+\frac{c}{ayt}\right)^{k}\exp\left(\frac{b\lambda xt}{(byt+d)}\right)L_{k}^{(A,\lambda)}\left(\frac{xyt}{(ayt+c)(byt+d)},y\right)\\&\quad=\sum_{n=0}^{\infty}\frac{n!}{k!}\left(-\frac{byt}{d}\right)^{n-k}L_{n}^{(A,\lambda)}(x,y)\{\Gamma(n-k+1)\}^{-1}\\&\quad\times{}_2F_1\left[-k,A+(n+1)I;n-k+1;\frac{bc}{ad}\right],\\&\quad\left|\frac{byt}{d}\right|<1;\quad\left|\frac{c}{ayt}\right|<1;\quad ad-bc=1;\quad n=0,1,2,\ldots.\end{aligned} \tag{3.17}$$

4. Representations of the Lie algebra $\mathcal{G}(0,1)$ and modified Laguerre matrix polynomials

Consider the irreducible representation $\uparrow_{\omega,\mu}$ of the harmonic oscillator Lie algebra $\mathcal{G}(0,1)$ [18; p 85] where $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum of this representation is the set $S = \{-\omega + n : n \text{ a nonnegative integer}\}$ and the representation space V has a basis $(W_n)_{n\in S}$ such that

$$\begin{aligned} &J^3 W_n = n W_n, \quad E W_n = \mu W_n, \quad J^+ W_n = \mu W_{n+1}, \\ &J^- W_n = (n+\omega) W_{n-1}, \quad C_{0,1} W_n = (J^+ J^- - E J^3) W_n = \mu\omega W_n, \end{aligned} \tag{4.1}$$

for all $n \in S$. The commutation relations satisfied by the operators J^+, J^-, J^3, E are

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = -E, \qquad [J^\pm, E] = [J^3, E] = 0. \tag{4.2}$$

In particular, we are looking for the functions $W_n(x,y,t) = Z_n(x,y)t^n$ such that equations (4.1) are satisfied for all $n \in S$. There are numerous possible solutions of equations (4.2). We consider the linear differential operators J^+, J^-, J^3, E of the following forms:

$$\begin{aligned} J^+ &= -y^2 t \frac{\partial}{\partial x} + \lambda y t, \\ J^- &= \frac{x}{\lambda y t}\frac{\partial}{\partial x} + \frac{1}{\lambda y}\frac{\partial}{\partial t} + \frac{1}{\lambda y t} A, \\ J^3 &= t\frac{\partial}{\partial t}, \\ E &= 1. \end{aligned} \tag{4.3}$$

The operators in equation (4.3) satisfy the commutation relations (4.2). In terms of the functions $Z_n(x,y)$ and using operators (4.3), relations (4.1) reduce to

$$\begin{aligned} &\text{(i)} \quad \left(-y^2\frac{\partial}{\partial x} + \lambda y\right) Z_n(x,y) = \mu Z_{n+1}(x,y), \\ &\text{(ii)} \quad \left(\frac{x}{\lambda y}\frac{\partial}{\partial x} + \frac{(A+nI)}{\lambda y}\right) Z_n(x,y) = (n+\omega) Z_{n-1}(x,y), \\ &\text{(iii)} \quad \left(-\frac{xy}{\lambda}\frac{\partial^2}{\partial x^2} - \left(\frac{yA}{\lambda} + \left(\frac{y}{\lambda}(n+1) - x\right) I\right)\frac{\partial}{\partial x} + A\right) Z_n(x,y) = \mu\omega Z_n(x,y). \end{aligned} \tag{4.4}$$

We can take $\omega = 0$ and $\mu = 1$, without any loss of generality. For this choice of ω and μ and in view of equations (2.26) and (2.27), we observe that

$$Z_n(x,y) = n!\left(\frac{x}{y}\right)^{(-A-nI)} f_n^{(A,\lambda)}(x,y),$$

satisfy equations (4.4).

Thus, we conclude that the matrix functions

$$W_n(x,y,t) = n!\left(\frac{x}{y}\right)^{(-A-nI)} f_n^{(A,\lambda)}(x,y) t^n,$$

where $n \in S$, form a basis for a realization of the representation $\uparrow_{0,1}$ of $\mathcal{G}(0,1)$.

This representation of $\mathcal{G}(0,1)$ can be extended to a local multiplier representation of the corresponding Lie group $G(0,1)$. Using operators (4.3), the local multiplier representation

$T(g)$, $g \in G(0, 1)$ defined on $\mathcal{F}$, the space of all functions analytic in a neighbourhood of the point $(x^0, y^0, t^0) = (1, 1, 1)$, takes the form

$$\begin{aligned}
&[T(\exp\tau\mathcal{J}^3)W](x, y, t) = W(x, y, te^\tau),\\
&[T(\exp c\mathcal{J}^-)W](x, y, t) = \left(1+\frac{c}{\lambda yt}\right)^A W\left(x\left(1+\frac{c}{\lambda yt}\right), y, t\left(1+\frac{c}{\lambda yt}\right)\right),\\
&[T(\exp b\mathcal{J}^+)W](x, y, t) = \exp(\lambda byt)W\left(x\left(1-\frac{by^2t}{x}\right), y, t\right),\\
&[T(\exp a\mathcal{E})W](x, y, t) = \exp(a)W(x, y, t).
\end{aligned} \tag{4.5}$$

For $g \in G(0, 1)$, we have

$$T(g) = T(\exp b\mathcal{J}^+)T(\exp c\mathcal{J}^-)T(\exp\tau\mathcal{J}^3)T(\exp a\mathcal{E})$$

and therefore we obtain

$$\begin{aligned}
[T(g)W](x, y, t) = &\left(1+\frac{c}{\lambda yt}\right)^A \exp(\lambda byt + a)\\
&\times W\left(x\left(1-\frac{by^2t}{x}\right)\left(1+\frac{c}{\lambda yt}\right), y, te^\tau\left(1+\frac{c}{\lambda yt}\right)\right).
\end{aligned} \tag{4.6}$$

The matrix elements of $T(g)$ with respect to the analytic basis $(W_n)_{n\in S}$ are the functions $A_{kn}(g)$ uniquely determined by $\uparrow_{0,1}$ of $\mathcal{G}(0, 1)$ and are defined by

$$[T(g)W_n](x, y, t) = \sum_{k=0}^{\infty} A_{kn}(g)W_k(x, y, t), \qquad n = 0, 1, 2, \ldots. \tag{4.7}$$

Therefore, we prove the following result.

Theorem 4.1. *The following generating equation holds:*

$$\begin{aligned}
&n!\exp(\lambda byt)\left(1-\frac{by^2t}{x}\right)^{(-A-nI)} f_n^{(A,\lambda)}\left(x\left(1-\frac{by^2t}{x}\right)\left(1+\frac{c}{\lambda yt}\right), y\right)\\
&\quad = \sum_{k=0}^{\infty} c^{n-k}k!\left(\frac{x}{y}\right)^{n-k} L_k^{(n-k)}(-bc)f_k^{(A,\lambda)}(x, y)t^{k-n},\\
&\quad \left|\frac{by^2t}{x}\right| < 1; \quad \left|\frac{c}{\lambda yt}\right| < 1; \quad n = 0, 1, 2, \ldots.
\end{aligned} \tag{4.8}$$

Proof. Using equations (4.6) and (4.7), we obtain

$$\begin{aligned}
&n!\left(\frac{x}{y}\right)^{-n}\left(1-\frac{by^2t}{x}\right)^{-A-nI} \exp(\lambda byt + \tau n + a)\ f_n^{(A,\lambda)}\left(x\left(1-\frac{by^2t}{x}\right)\left(1+\frac{c}{\lambda yt}\right), y\right)t^n\\
&\quad = \sum_{k=0}^{\infty} A_{kn}(g)k!\left(\frac{x}{y}\right)^{-k} f_n^{(A,\lambda)}(x, y)t^k,\\
&\quad \left|\frac{by^2t}{x}\right| < 1; \quad \left|\frac{c}{\lambda yt}\right| < 1; \quad n = 0, 1, 2, \ldots
\end{aligned} \tag{4.9}$$

and the matrix elements $A_{kn}(g)$ are given by [18; p 87(4.26)], (for $\omega = 0$ and $\mu = 1$),

$$A_{kn}(g) = \exp((a + n\tau)c^{n-k}L_k^{(n-k)}(-bc), \qquad k, n \geqslant 0. \tag{4.10}$$

Substituting the value of $A_{kn}(g)$ given by equation (4.10) into equation (4.9) and simplifying, we obtain result (4.8).

Next, we consider the irreducible representation $\downarrow_{\omega,\mu}$ of the Lie algebra $\mathcal{G}(0,1)$ [18; p 89], where $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum of this representation is the set $S = \{-\omega - 1 - n : n$ a nonnegative integer$\}$ and the representation space V has a basis $(W_n)_{n\in S}$ such that

$$\begin{aligned} &J^3 W_n = n W_n, \quad E W_n = -\mu W_n, \quad J^+ W_n = -(n+\omega+1) W_{n+1}, \\ &J^- W_n = \mu W_{n-1}, \quad C_{0,1} W_n = (J^+J^- - EJ^3) W_n = -\mu\omega W_n, \end{aligned} \tag{4.11}$$

for all $n \in S$, the commutation relations satisfied by the operators J^+, J^-, J^3, E are same as relations (4.2). In particular, we are looking for the functions $W_n(x, y, t) = Z_n(x, y)t^n$ such that relations (4.11) are satisfied for all $n \in S$.

Now we assume that the linear differential operators J^+, J^-, J^3, E take the following forms:

$$\begin{aligned} &J^+ = xyt\frac{\partial}{\partial x} - yt^2\frac{\partial}{\partial t} - (\lambda x I + yA)t, \\ &J^- = -\frac{1}{\lambda t}\frac{\partial}{\partial x}, \\ &J^3 = t\frac{\partial}{\partial t}, \\ &E = 1. \end{aligned} \tag{4.12}$$

The operators in equations (4.12) satisfy the commutation relations (4.2). In terms of the functions $Z_n(x, y)$ and using operators (4.12), relations (4.11) reduce to

$$\begin{aligned} &\text{(i)} \quad \left(xy\frac{\partial}{\partial x} - (yA + (\lambda x + ny)I)\right) Z_n(x, y) = -(n+\omega+1) Z_{n+1}(x, y), \\ &\text{(ii)} \quad -\frac{1}{\lambda}\frac{\partial}{\partial x} Z_n(x, y) = \mu Z_{n-1}(x, y), \\ &\text{(iii)} \quad -\frac{1}{\lambda}\left(xy\frac{\partial^2}{\partial x^2} - (yA + (\lambda x - y(1-n))I)\frac{\partial}{\partial x} + \lambda n\right) Z_n(x, y) = -\mu\omega Z_n(x, y). \end{aligned} \tag{4.13}$$

We can take $\omega = 0$ and $\mu = -1$, without any loss of generality. For this choice of ω and μ, we observe that (i) and (ii) of equations (4.13) agree with the first two recurrence relations of equation (2.26) and (iii) of equation (4.13) coincides with the matrix differential equation (2.27) of 2VMLM$_a$P $f_n^{(A,\lambda)}(x, y)$. In fact, for all $n \in S$ the choice for $Z_n(x, y) = f_n^{(A,\lambda)}(x, y)$ satisfies equations (4.13).

Thus, we conclude that the functions

$$W_n(x, y, t) = f_n^{(A,\lambda)}(x, y)t^n, \qquad n \in S$$

form a basis for a realization of the representation $\downarrow_{0,-1}$ of $\mathcal{G}(0,1)$.

For the operators (4.12), the local multiplier representation $T'(g)$, $g \in G(0,1)$ defined on $\mathcal{F}$, the space of all functions analytic in a neighbourhood of the point $(x^0, y^0, t^0) = (1, 1, 1)$, takes the form

$$[T'(g)W](x, y, t) = (1 + byt)^{-A} \exp(-\lambda bxt + a) W\left(x(1+byt)\left(1 - \frac{c}{\lambda xt}\right), y, \frac{te^{\tau}}{(1+byt)}\right). \tag{4.14}$$

The matrix elements of $T'(g)$ with respect to the analytic basis $(W_n)_{n\in S}$ are the functions $B_{kn}(g)$ uniquely determined by $\downarrow_{0,-1}$ of $\mathcal{G}(0,1)$ and are defined by

$$[T(g)W_n](x,y,t) = \sum_{k=0}^{\infty} B_{kn}(g) W_k(x,y,t), \qquad n = 0,1,2,\ldots. \tag{4.15}$$

□

Therefore, we prove the following result.

Theorem 3. *The following generating equation holds:*

$$(1+byt)^{(-A-nI)} \exp(-\lambda bxt + (2n+1)\tau + bc)\ f_n^{(A,\lambda)}\left(x(1+byt)\left(1-\frac{c}{\lambda xt}\right), y\right)$$
$$= \sum_{k=0}^{\infty} b^{n-k} L_k^{n-k}(bc)\ f_k^{(A,\lambda)}(x,y) t^{k-n},$$
$$|byt| < 1; \quad \left|\frac{c}{\lambda xt}\right| < 1; \quad n = 0,1,2,\ldots. \tag{4.16}$$

Proof. Using equations (4.14) and (4.15), we obtain

$$(1+byt)^{(-A-nI)} \exp(-\lambda bxt + \tau n + a)\ f_n^{(A,\lambda)}\left(x(1+byt)\left(1-\frac{c}{\lambda yt}\right), y\right) t^n$$
$$= \sum_{k=0}^{\infty} B_{kn}(g)\ f_k^{(A,\lambda)}(x,y) t^k,$$
$$|byt| < 1; \quad \left|\frac{c}{\lambda xt}\right| < 1; \quad n = 0,1,2,\ldots. \tag{4.17}$$

and the matrix elements $B_{kn}(g)$ are given by [18; p 91(4.36)](for $\omega = 0$ and $\mu = -1$)

$$B_{kn}(g) = \exp(-bc - (n+1)\tau + a) b^{n-k} L_k^{(n-k)}(bc), \qquad k,n \geqslant 0. \tag{4.18}$$

Substituting the value of $B_{kn}(g)$ given by equation (4.18) into equation (4.17) and simplifying, we obtain result (4.16). □

5. Special cases

First, we note the following special cases of the 2VLM$_a$P $L_n^{(A,\lambda)}(x,y)$.

(1) For $A = \alpha \in \mathbb{C}^{1\times 1}$ and $\lambda = I = 1$, we have

$$L_n^{(\alpha,1)}(x,y) = L_n^{(\alpha)}(x,y), \tag{5.1}$$

where $L_n^{(\alpha)}(x,y)$ are the 2VALP defined by equations (1.11) and (1.12).

(2) For $y = 1$, we have

$$L_n^{(A,\lambda)}(x,1) = L_n^{(A,\lambda)}(x), \tag{5.2}$$

where $L_n^{(A,\lambda)}(x)$ are the LM$_a$P defined by equations (1.4) and (1.6).

(3) For $A = \alpha \in \mathbb{C}^{1\times 1}$, $\lambda = I = y = 1$, we have

$$L_n^{(\alpha,1)}(x,1) = L_n^{(\alpha)}(x). \tag{5.3}$$

(4) For $y = 0$, we have

$$L_n^{(A,\lambda)}(x,0) = \frac{(-1)^n \lambda^n x^n I}{n!} \tag{5.4}$$

and for $x = 0$, we have

$$L_n^{(A,\lambda)}(0,y) = \frac{(A+I)_n y^n}{n!}. \tag{5.5}$$

Also, we note the following special cases of the $2\text{VMLM}_\text{a}\text{P}$ $f_n^{(A,\lambda)}(x, y)$:

(i) For $A = \beta \in \mathbb{C}^{1\times 1}$ and $\lambda = 1$, we have

$$f_n^{(\beta,1)}(x, y) = f_n^{\beta}(x, y), \tag{5.6}$$

where $f_n^{\beta}(x, y)$ are the 2VMLP defined by equations (2.18) and (2.19).

(ii) For $y = 1$, we have

$$f_n^{(A,\lambda)}(x, 1) = f_n^{(A,\lambda)}(x), \tag{5.7}$$

where $f_n^{(A,\lambda)}(x)$ denotes the modified Laguerre matrix polynomials ($\text{MLM}_\text{a}\text{P}$) defined by the generating function

$$(1 - t)^{-A} \exp(\lambda x t) = \sum_{n=0}^{\infty} f_n^{(A,\lambda)}(x) t^n, \tag{5.8}$$

and specified by the series

$$f_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(A)_{n-k} (\lambda x)^k}{k!(n - k)!}, \qquad n = 0, 1, 2, \ldots. \tag{5.9}$$

(iii) For $A = \beta \in \mathbb{C}^{1\times 1}$ and $y = \lambda = 1$, we have

$$f_n^{(\beta,1)}(x, 1) = f_n^{\beta}(x), \tag{5.10}$$

where $f_n^{\beta}(x)$ denotes the modified Laguerre polynomials (MLP) [17; p 9].

(iv) For $y = 0$, we obtain

$$f_n^{(A,\lambda)}(x, 0) = \frac{\lambda^n x^n I}{n!} \tag{5.11}$$

and for $x = 0$, we obtain

$$f_n^{(A,\lambda)}(0, y) = \frac{(A)_n y^n}{n!}. \tag{5.12}$$

In view of the above-mentioned special cases we drive the generating relations involving the polynomials related to $2\text{VLM}_\text{a}\text{P}$ $L_n^{(A,\lambda)}(x, y)$ and $2\text{VMLM}_\text{a}\text{P}$ $f_n^{(A,\lambda)}(x, y)$ by taking suitable values to the parameters and variables in generating relations (3.14), (3.17) and (4.8).

(1) Taking $A = \alpha \in \mathbb{C}^{1\times 1}$ and $\lambda = I = 1$ in equations (3.14) and (3.17) and using equation (5.1), we obtain

$$\begin{aligned}
&\left(1 + \frac{byt}{d}\right)^{-(\alpha+\nu+1)} \left(1 + \frac{c}{ayt}\right)^{\nu} \exp\left(\frac{bxt}{(byt + d)}\right) L_\nu^{(\alpha)}\left(\frac{xyt}{(ayt + c)(byt + d)}, y\right) \\
&\quad = \sum_{n=-\infty}^{\infty} \frac{\Gamma(1 + \nu + n)}{\Gamma(1 + \nu)} \left(\frac{-byt}{d}\right)^n L_{n+\nu}^{(\alpha)}(x, y)\{\Gamma(1 + n)\}^{-1} \\
&\quad \times {}_2F_1\left[-\nu, \alpha + \nu + n + 1; n + 1; \frac{bc}{ad}\right], \\
&\quad \left|\frac{byt}{d}\right| < 1; \quad \left|\frac{c}{ayt}\right| < 1; \quad ad - bc = 1
\end{aligned} \tag{5.13}$$

and

$$\left(1+\frac{byt}{d}\right)^{-(\alpha+k+1)}\left(1+\frac{c}{ayt}\right)^{k}\exp\left(\frac{bxt}{(byt+d)}\right)L_k^{(\alpha)}\left(\frac{xyt}{(ayt+c)(byt+d)},y\right)$$
$$=\sum_{n=0}^{\infty}\frac{n!}{k!}\left(\frac{-byt}{d}\right)^{n-k}L_n^{(\alpha)}(x,y)\{\Gamma(n-k+1)\}^{-1}$$
$$\times{}_2F_1\left[-k,\alpha+n+1;n-k+1;\frac{bc}{ad}\right],$$
$$\left|\frac{byt}{d}\right|<1;\quad\left|\frac{c}{ayt}\right|<1;\quad ad-bc=1, \tag{5.14}$$

(2) Taking $y=1$ in equations (3.14) and (3.17) and using equation (5.2), we obtain

$$\left(1+\frac{bt}{d}\right)^{-(A+(\nu+1)I)}\left(1+\frac{c}{at}\right)^{\nu}\exp\left(\frac{b\lambda xt}{(bt+d)}\right)L_\nu^{(A,\lambda)}\left(\frac{xt}{(at+c)(bt+d)}\right)$$
$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+\nu+n)}{\Gamma(1+\nu)}\left(\frac{-bt}{d}\right)^{n}L_{n+\nu}^{(A,\lambda)}(x)\{\Gamma(1+n)\}^{-1}$$
$$\times{}_2F_1\left[-\nu,A+(\nu+n+1)I;n+1;\frac{bc}{ad}\right],$$
$$\left|\frac{bt}{d}\right|<1;\quad\left|\frac{c}{at}\right|<1;\quad ad-bc=1 \tag{5.15}$$

and

$$\left(1+\frac{bt}{d}\right)^{-(A+(k+1)I)}\left(1+\frac{c}{at}\right)^{k}\exp\left(\frac{b\lambda xt}{(bt+d)}\right)L_k^{(A,\lambda)}\left(\frac{xt}{(at+c)(bt+d)}\right)$$
$$=\sum_{n=0}^{\infty}\frac{n!}{k!}\left(\frac{-bt}{d}\right)^{n-k}L_n^{(A,\lambda)}(x)\{\Gamma(n-k+1)\}^{-1}$$
$$\times{}_2F_1\left[-k,A+(n+1)I;n-k+1;\frac{bc}{ad}\right],$$
$$\left|\frac{bt}{d}\right|<1;\quad\left|\frac{c}{at}\right|<1;\quad ad-bc=1, \tag{5.16}$$

respectively.

Also, taking $A=\alpha\in\mathbb{C}^{1\times1}$ and $y=\lambda=I=1$ in equations (3.14) and (3.17) and using equation (5.3), we obtain [22; pp. 330–331 (29) and (32)].

(3) Taking $a=d=t=1$, $c=0$ and replacing b by $-b$ in equation (3.14), we obtain

$$(1-by)^{-(A+(\nu+1)I)}\exp\left(\frac{-b\lambda x}{(1-by)}\right)L_\nu^{(A,\lambda)}\left(\frac{xy}{(1-by)},y\right)$$
$$=\sum_{n=0}^{\infty}\frac{(1+\nu)_n}{n!}L_{n+\nu}^{(A,\lambda)}(x,y)(by)^n,\qquad |by|<1. \tag{5.17}$$

Further, taking $A = \alpha \in \mathbb{C}^{1\times 1}$ and $\lambda = I = 1$ in equation (5.17), we obtain

$$(1-by)^{-(\alpha+\nu+1)} \exp\left(\frac{-bx}{(1-by)}\right) L_\nu^{(\alpha)}\left(\frac{xy}{(1-by)}, y\right) = \sum_{n=0}^{\infty} \frac{(1+\nu)_n}{n!} L_{n+\nu}^{(\alpha)}(x,y)(by)^n, \qquad |by| < 1, \tag{5.18}$$

which for $y = 1$ reduces to the result [22; p 331(30)].

(4) Taking $A = \beta \in \mathbb{C}^{1\times 1}$ and $\lambda = I = 1$ in equation (4.8) and using equation (5.6), we obtain

$$n!\exp(byt)\left(1-\frac{by^2t}{x}\right)^{(-\beta-n)} f_n^\beta\left(x\left(1-\frac{by^2t}{x}\right)\left(1+\frac{c}{yt}\right), y\right) = \sum_{k=0}^{\infty} c^{n-k}\left(\frac{x}{y}\right)^{n-k} \times k! L_k^{(n-k)}(-bc) f_k^\beta(x,y) t^{k-n},$$
$$\left|\frac{by^2t}{x}\right| < 1; \qquad \left|\frac{c}{yt}\right| < 1; \qquad n = 0, 1, 2, \ldots. \tag{5.19}$$

Further, replacing β by $-p$, t by $-t$ in equation (5.19) and then using the relation

$$f_n^\beta(x,y) = (-1)^n L_n^{(-\beta-n)}(x,y), \tag{5.20}$$

we obtain

$$n!\exp(-byt)\left(1+\frac{by^2t}{x}\right)^{p-n} L_n^{(p-n)}\left(x\left(1+\frac{by^2t}{x}\right)\left(1-\frac{c}{yt}\right), y\right) = \sum_{k=0}^{\infty} c^{n-k} k!\left(\frac{x}{y}\right)^{n-k} L_k^{(n-k)}(-bc) L_k^{(p-k)}(x,y) t^{k-n},$$
$$\left|\frac{by^2t}{x}\right| < 1; \qquad \left|\frac{c}{yt}\right| < 1, \tag{5.21}$$

which for $y = 1$, reduces to the result [18; p 112(4.94)].

(5) Taking $b = 0$ in equation (4.8) and making use of the limit [18; p 88(4.29)], we obtain

$$f_n^{(A,\lambda)}\left(x\left(1+\frac{c}{\lambda yt}\right), y\right) = \sum_{k=0}^{\infty} \frac{1}{(n-k)!}\left(\frac{cx}{yt}\right)^{n-k} f_k^{(A,\lambda)}(x,y), \qquad \left|\frac{c}{\lambda yt}\right| < 1. \tag{5.22}$$

Further, taking $A = \beta \in \mathbb{C}^{1\times 1}$, $\lambda = 1$ and replacing t by x and k by $n-k$ in equation (5.22) and then using equation (5.6), we obtain

$$f_n^\beta\left(x\left(1+\frac{c}{xy}\right), y\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{c}{y}\right)^k f_{n-k}^\beta(x,y), \qquad \left|\frac{c}{xy}\right| < 1, \tag{5.23}$$

which for $y = 1$, reduces to the result [17; p 45(7)].

Again, replacing β by $-p$, c by $-c$ in equation (5.23) and then using the relation (5.20) we obtain

$$L_n^{(p-n)}\left(x\left(1-\frac{c}{xy}\right), y\right) = \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{c}{y}\right)^k L_{n-k}^{(p-n+k)}(x,y), \qquad \left|\frac{c}{xy}\right| < 1, \tag{5.24}$$

which on taking $y = 1$ and $p - n = q$ reduces to the result [18; p 113].

(6) Taking $c = 0$ in equation (4.8) and making use of the limit [18; p 88(4.29)], we obtain

$$\exp(\lambda byt)\left(1 - \frac{by^2t}{x}\right)^{(-A-nI)} f_n^{(A,\lambda)}\left(x\left(1 - \frac{by^2t}{x}\right), y\right)$$
$$= \sum_{k=0}^{\infty} \binom{k}{n} \left(\frac{byt}{x}\right)^{k-n} f_k^{(A,\lambda)}(x, y), \qquad \left|\frac{by^2t}{x}\right| < 1. \tag{5.25}$$

Further, taking $A = -p$, $\lambda = I = 1$ and replacing t by x, b by $-b$ and k by k+n and using relation (5.20), we obtain

$$\exp(-byt)(1 + by^2)^{p-n} L_n^{(p-n)}(x(1 + by^2), y)$$
$$= \sum_{k=0}^{\infty} \binom{k+n}{n} (by)^k L_{k+n}^{(p-k-n)}(x, y), \qquad |by^2| < 1, \tag{5.26}$$

which on taking $y = 1$ and $p - n = q$ reduces to the result [18; p 112].

For the same choices of parameters and variables we can obtain several other new results corresponding to generating relation (4.16).

6. Concluding remarks

We have derived the generating relations involving $2\mathrm{VLM_aP}$ $L_n^{(A,\lambda)}(x, y)$ by constructing a three-dimensional Lie algebra isomorphic to the special linear algebra $sl(2)$, using Weisner's method [23]. We have also derived generating relations involving $2\mathrm{VMLM_aP}$ $f_n^{(A,\lambda)}(x, y)$ by using the representations $\uparrow_{\omega,\mu}$ and $\downarrow_{\omega,\mu}$ of the Lie algebra $\mathcal{G}(0, 1)$, respectively.

In this section, we consider the $2\mathrm{VLM_aP}$ $L_n^{(A,\lambda)}(x, y)$ within the Lie algebra representation formalism. We discuss the problem of framing these polynomials into the context of the representation $D(u, m_0)$ [18; p 184] of the special linear algebra $sl(2)$. We extend the realization of the representation $D(u, m_0)$ of the Lie algebra $sl(2)$ to a local multiplier representation of the corresponding Lie group $SL(2)$. For this purpose, we consider the following convenient form of $2\mathrm{VLM_aP}$ $L_n^{(A,\lambda)}(x, y)$:

$$(1 - yt)^{-2(u+1)I} \exp\left(\frac{-\lambda xt}{1 - yt}\right) = \sum_{n=0}^{\infty} L_{n-u-1}^{((2u+1)I,\lambda)}(x, y)t^n, \tag{6.1}$$

where $(n - u)$ is not an integer.

The irreducible representation $D(u, m_0)$ of $sl(2)$ is defined for $u, m_0 \in \mathbb{C}$ such that $0 \leqslant \mathrm{Re}(m_0) < 1$ and $u \pm m_0$ are not integers. The spectrum of this representation is the set $S = \{m_0 + n : n \text{ an integer}\}$. There is a basis $\{f_m : m \in S\}$ for the representation space V such that

$$\begin{aligned} &J^3 f_m = m f_m, \quad J^+ f_m = (m - u) f_{m+1}, \quad J^- f_m = -(m + u) f_{m-1}, \\ &C_{1,0} f_m = (J^+J^- + J^3J^3 - J^3) f_m = u(u + 1) f_m, \end{aligned} \tag{6.2}$$

for all $m \in S$. These operators satisfy the commutation relations (3.4).

In order to find a realization of this representation, we look for the functions

$$f_m(x, y; t) = Z_m(x, y)t^m,$$

in $\mathcal{F}$, the space of all functions analytic in a neighbourhood of the point $(x^0, y^0, t^0) = (1, 1, 0)$ such that relations (6.2) are satisfied for all $m \in S$.

We take the set of linear differential operators K^3, K^+, K^- as follows:

$$\begin{aligned} K^3 &= t\frac{\partial}{\partial t},\\ K^+ &= xyt\frac{\partial}{\partial x} + yt^2\frac{\partial}{\partial t} + (y(u+1) - \lambda x)t,\\ K^- &= \frac{x}{yt}\frac{\partial}{\partial x} - \frac{1}{y}\frac{\partial}{\partial t} + \frac{(u+1)}{yt}. \end{aligned} \tag{6.3}$$

The operators in equation (6.3) satisfy the commutation relations (3.4). In terms of the functions $Z_m(x, y)$ and using operators (6.3), relations (6.2) reduce to

$$\begin{aligned} &\text{(i)} \quad \left(xy\frac{\partial}{\partial x} + (m+u+1)y - \lambda x\right)Z_m(x, y) = (m-u)Z_{m+1}(x, y),\\ &\text{(ii)} \quad \left(\frac{x}{y}\frac{\partial}{\partial x} - \frac{1}{y}(m-u-1)\right)Z_m(x, y) = -(m+u)Z_{m-1}(x, y),\\ &\text{(iii)} \quad \left(x\frac{\partial^2}{\partial x^2} + \left(2(u+1) - \frac{\lambda x}{y}\right)\frac{\partial}{\partial x} + \frac{\lambda}{y}(m-u-1)\right)Z_m(x, y) = 0. \end{aligned} \tag{6.4}$$

Without any loss of generality, we can choose $m, u \in \mathbb{C}$ such that $(m - u)$ is not an integer. For this choice of m, u and for all $m \in S$ the functions

$$Z_m(x, y) = L_{m-u-1}^{((2u+1)I,\lambda)}(x, y)$$

satisfy relations (6.4).

Thus we conclude that, the functions

$$f_m(x, y, t) = L_{m-u-1}^{((2u+1)I,\lambda)}(x, y)t^m,$$

$m \in S$ form a basis for a realization of the representation $D(u, m_0)$ of $sl(2)$. In the usual manner this realization can be extended to a local multiplier representation T of $SL(2)$ on the space $\mathcal{F}$. Using operators (6.3), the local multiplier representation takes the form

$$\begin{aligned} &[T(\exp\tau'\mathcal{J}^3)f](x, y, t) = f(x, y, te^{\tau'}),\\ &[T(\exp c'\mathcal{J}^-)f](x, y, t) = \left(1 - \frac{c'}{yt}\right)^{u+1} f\left(\frac{x}{\left(1-\frac{c'}{yt}\right)}, y, t\left(1 - \frac{c'}{yt}\right)\right), \qquad \left|\frac{c'}{yt}\right| < 1,\\ &[T(\exp b'\mathcal{J}^+)f](x, y, t) = (1 - b'yt)^{-u-1}\\ &\qquad \times \exp\left(\frac{-b'\lambda xt}{(1 - b'yt)}\right) f\left(\frac{x}{(1 - b'yt)}, y, \frac{t}{(1 - b'yt)}\right), \qquad |b'yt| < 1, \end{aligned} \tag{6.5}$$

valid for all $f \in \mathcal{F}$, (x, y, t) in the domain of $\mathcal{F}$ and for $|b'|$, $|c'|$ and $|\tau'|$ sufficiently small. Therefore, the operator $T(g)$ is given by

$$\begin{aligned} [T(g)f](x, y, t) &= [T(\exp b'\mathcal{J}^+)T(\exp c'\mathcal{J}^-)T(\exp\tau'\mathcal{J}^3)f](x, y, t)\\ &= (1 - b'yt)^{-u-1}\exp\left(\frac{-b'\lambda xt}{(1 - b'yt)}\right)\left(\frac{yt}{(yt - c' + b'c'yt)}\right)^{u+1}\\ &\quad \times f\left(\frac{xyt}{(1 - b'yt)(yt - c' + b'c'yt)}, y, \frac{(yt - c' + b'c'yt)e^{\tau'}}{y(1 - b'yt)}\right), \end{aligned} \tag{6.6}$$

which after setting $b' = -\frac{b}{d}$, $c' = -cd$, $\exp\left(\frac{\tau'}{2}\right) = d^{-1}$ and using the fact that $ad - bc = 1$ gives us

$$[T(g)f](x, y, t) = (byt + d)^{-u-1}\left(a + \frac{c}{yt}\right)^{-u-1} \exp\left(\frac{b\lambda xt}{(byt + d)}\right)$$
$$\times f\left(\frac{xyt}{(ayt + c)(byt + d)}, y, \frac{(ayt + c)}{y(byt + d)}\right),$$
$$\left|\frac{c}{ayt}\right| < 1; \quad \left|\frac{byt}{d}\right| < 1; \quad ad - bc = 1, \tag{6.7}$$

for $f \in \mathcal{F}$ and g in a small enough neighbourhood of e so that the above expression is uniquely defined. The matrix elements of $T(g)$ with respect to the analytic basis $(f_m)_{m\in S}$ are the functions $A_{lk}(g)$ uniquely determined by $D(u, m_0)$ of $sl(2)$ and are defined by

$$[T(g)f_{m_0+k}](x, y, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(x, y, t), \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{6.8}$$

Now, using equations (6.7) and (6.8), we obtain

$$(1 + bc)^{-u-1-m_0-k}\left(1 + \frac{byt}{d}\right)^{-u-1-m_0-k}\left(1 + \frac{c}{ayt}\right)^{-u-1+m_0+k} a^{2(m_0+k)} t^{m_0+k} \exp\left(\frac{b\lambda xt}{(byt + d)}\right)$$
$$\times L^{((2u+1)I,\lambda)}_{m_0+k-u-1}\left(\frac{x}{\left(1 + \frac{c}{ayt}\right)(1 + bc)\left(1 + \frac{byt}{d}\right)}, y\right)$$
$$= \sum_{l=-\infty}^{\infty} A_{lk}(g) L^{((2u+1)I,\lambda)}_{m_0+l-u-1}(x, y) t^{m_0+l},$$

which further simplifies to

$$(1 + bc)^{-\nu-\mu-1}\left(1 + \frac{byt}{d}\right)^{-\nu-\mu-1}\left(1 + \frac{c}{ayt}\right)^{\nu} \exp\left(\frac{b\lambda xt}{(byt + d)}\right) a^{2\nu+\mu+1}$$
$$\times L^{(\mu I,\lambda)}_{\nu}\left(\frac{x}{\left(1 + \frac{c}{ayt}\right)(1 + bc)\left(1 + \frac{byt}{d}\right)}, y\right) = \sum_{l=-\infty}^{\infty} A_{lk}(g) L^{(\mu I,\lambda)}_{\nu-l}(x, y) t^{-l},$$
$$|bc| < 1; \quad \left|\frac{c}{a}\right| < |yt| < \left|\frac{d}{b}\right|; \quad -\pi < \arg(a), \arg(d) < \pi; \quad ad - bc = 1. \tag{6.9}$$

An explicit expression for the matrix elements $A_{lk}(g)$ is

$$A_{lk}(g) = \frac{(1 + bc)^{-\nu-1} a^{2\nu+\mu+1-l} c^l \Gamma(\mu + \nu + 1)}{\Gamma(l + 1)\Gamma(\mu + \nu - l + 1)}\, {}_2F_1\left[-\mu - \nu + l, \nu + 1; l + 1; \frac{bc}{ad}\right], \tag{6.10}$$

where ${}_2F_1$ is the hypergeometric function [1]. Substituting the value of $A_{lk}(g)$ given by equation (6.10) into equation (6.9) and simplifying we obtain the following generating equation:

$$(1+bc)^{-\mu}\left(1+\frac{byt}{d}\right)^{-\nu-\mu-1}\left(1+\frac{c}{ayt}\right)^{\nu}\exp\left(\frac{b\lambda xt}{(byt+d)}\right)$$
$$\times L_{\nu}^{(\mu I,\lambda)}\left(\frac{x}{\left(1+\frac{c}{ayt}\right)(1+bc)\left(1+\frac{byt}{d}\right)},y\right)=\sum_{l=-\infty}^{\infty}\frac{\left(\frac{c}{at}\right)^{l}\Gamma(\mu+\nu+1)}{\Gamma(\mu+\nu-l+1)\Gamma(l+1)}$$
$$\times {}_2F_1\left[-\mu-\nu+l,\nu+1;l+1;\frac{bc}{ad}\right]L_{\nu-l}^{(\mu I,\lambda)}(x,y),$$
$$|bc|<1;\quad \left|\frac{c}{ay}\right|<t<\left|\frac{d}{by}\right|;\quad d=\frac{1+bc}{a}, \tag{6.11}$$

where $\mu,\nu\in\mathbb{C}$ such that ν and $\mu+\nu$ are not integers.

In equation (6.11), when $l+1\leqslant 0$, the hypergeometric function ${}_2F_1$ is defined by

$$\lim_{c\to -n}\frac{{}_2F_1[a,b;c;t]}{\Gamma(c)}=\frac{a(a+1)\ldots(a+n)b(b+1)\ldots(b+n)t^{n+1}}{(n+1)!}$$
$$\times {}_2F_1[a+n+1,b+n+1;n+2;t],\qquad n=0,1,2,\ldots. \tag{6.12}$$

In general, the right-hand side of relation (6.11) converges whenever the left-hand side does.

In generating relation (6.11) taking $\lambda=I=1$, we obtain the scalar result [16; p 64(15)] which for $y=1$ reduces to the result [18; p 187(5.94)].

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